Fluids

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Books

Several books are listed in the printed notes for this course. Of these, the best are M van Dyke's "An album of fluid motion", which illustrates our results with wonderful pictures of actual fluids, and DJ Acheson's "Elemental Fluid Dynamics". It is also very much worth looking at G M Homsy et al's "Multimedia Fluid Mechanics", and, finally, the pictures and course notes at www.damtp.cam.ac.uk/user/ngb23/FD.

The reader should ensure they are familiar with last year's Vector Calculus course before starting this one.

We shall use $[f]$ to denote the dimensions of f, and $\equiv$ to mean equality of dimensions. Recall that $\vec{u} \cdot \vec{\nabla}$ represents a directional derivative in the direction of $\vec{u}$.

The definition of a fluid is in some sense the most important part of this course, and there is no truly good definition. The critical difference between a fluid and a solid is that a fluid cannot resist the forces applied to it - one way of describing this is to say that fluids cannot support shear stress when at rest - but of course we are only really interested in fluids when they are moving. It is not always obvious whether something is a fluid, as e.g. "silly putty" behaves as a fluid on long timescales but not short ones.

Two important ideas in fluids

Fluids contain many molecules. Some physicists will therefore attempt to model them by considering each molecule individialy, but this is impractical to calculate. We will instead consider "packets" of fluid, large enough that the particulate nature of the fluid is not apparent, but small enough that the packet can be treated as having a single velocity, which is simply the average velocity over the small volume of the packet.

There are two ways to view a fluid. In Lagrangian mechanics, as is usually used with systems of solids, we consider specific packets, and follow their position and velocity over time. However, simpler and generally more useful for fluids is the Eulerian method where we consider fixed points and trace the velocity there over time as the specific packet changes; we shall be using this method exclusively during this course, which will initially give some confusing results.

There are three main ways of visualizing fluid flows, which are equivalent if the flow does not change over time. A pathline is the trajectory of a single fluid packet over time. Streaklines are locus of positions (at the end of the time interval) of all the packets passing through a fixed point in a given time interval. Finally a streamline is a sort of instantaneous pathline, intuitively made by composing infinitesimal instantaneous streaklines at a fixed time.

We use $\vec{x}$ to mean the vector (x, y) and similar. For a flow $\vec{u}(\vec{x}, t)$ and example $(yt, 1)$:

Pathlines are curves $\vec{X}(t)$ found by solving $\frac{\partial}{\partial t}\vec{X}(t) = \vec{u}\left(\vec{X}(t), t\right)$, with boundary condition $\vec{X} = \vec{x_0}$ when $t = 0$; in our example we find (eliminating t from the result to find the line in terms of x and y) $X = x_0 + \frac{1}{2}y_0(Y - y_0)^2 + \frac{1}{3}(Y - y_0)^3$

Streaklines are curves $\vec{X}(s, x_0, t)$ where s is an arclength parameter along the ltceakline found by solving $\frac{\partial \vec{X}}{\partial t} = \vec{u}\left(\vec{X}, t\right)$ with $\vec{X} = \vec{x_0}$ when $s = t$. In this case we eliminate s to obtain the streakline as a function of t; in our example we have $X = \frac{1}{2}tY^2 - \frac{1}{6}Y^3$; note that this is different from the pathline

Streamlines are again curves $\vec{X}(s, x_0, t)$ but with $\frac{\partial \vec{X}}{\partial s} = \vec{u}(\vec{s}, t)$, and $\vec{X} = \vec{x_0}$ at $s = 0$. In our example, eliminating s we have $X = x_0 + \frac{1}{2}\left(Y^2 - y_0^2\right)t$ which is different again. However, in part IB these curves are generally coincident (this is the case precisely if the flow is steady); in this case streamlines are generally the easiest to calculate.

Material Derivative

This is the rate of change with time as seen when following a particular fluid parcel. We derive this derivative by $\delta f = f(\vec{x} + \delta\vec{x}, t + \delta t) - f(\vec{x}, t) = \delta\vec{x} \cdot \nabla f + \frac{\partial f}{\partial t}\delta t$ (plus higher order terms which disappear in the limit); the first term here is the spatial gradient of f and the second is the temporal gradient at constant $\vec{x}$. We have $\delta\vec{x} = \vec{u}(\vec{x}, t)\,\delta t$, so $\delta f = \left((\vec{u} \cdot \nabla) f + \frac{\partial f}{\partial t}\right)\delta t$; Taking the limit what in fluid mechanics is called $\frac{Df}{Dt}$ is $\frac{\partial f}{\partial t} + \vec{u} \cdot \nabla f$; we sometimes write this as $\frac{D}{Dt} = \frac{\partial}{\partial t} + \vec{u} \cdot \nabla$. Compare this with the chain rule for differentiation.

This is sometimes called the Lagrangian derivative though this is misleading. It is the sum of the Eulerian temporal derivative and what is called the advected or convected derivative.

Kinematic boundary cond

We have a first order DE so need one boundary cond; there is no mass change at the boundary i.e. no normal flow, $\vec{u} \cdot \vec{n} = 0$; while the tangential flow is important we will not cover it in this course. If the boundary moves with vel

$\vec{U}(\vec{x},t)$ then there is no normal frow in its rest frame, i.e. $\vec{u}\cdot\vec{n}=\vec{U}\cdot\vec{n}$ on the boundary curve c.

Incompressibility

In general $\frac{D\rho}{Dt}+\rho\nabla\cdot\vec{u}=0$

In IB we have ρ constant so $\nabla\cdot\vec{u}=0$; this is obviously false IRL. Most immediately this excludes sound, suggesting it is only valid for $|\vec{u}|<<c$ the speed of sound; note that c is very small in e.g. air-water mixtures. c for a given substance is found by $c^2=\frac{\partial p}{\partial\rho}\mid_S$, the change of pressure with density where entropy is constant. With this restriction we can still model most water flows e.g. rivers, water waves, subsonic aircraft, bubble motion, and incident jets.

Stream Functions

These are 2D flows, say varying only in x,y. We can express them in 2D cartesian coordinates $(u,v,0)$ with $\frac{\partial}{\partial z}\equiv 0$. Recall as in VC that for $\vec{u}=(P,Q,0)$ as $\nabla\cdot\vec{u}=0$ we must have $\frac{\partial P}{\partial y}=\frac{\partial Q}{\partial x}$, and then we have $P=\frac{\partial\psi}{\partial x}$ etc. for some potential ψ obtainable by integration. $\psi=c$ for constant c gives the streamlines, tangential to $\vec{u}$. Where $\vec{\nabla}\psi$ is larger, i.e. the streamlines are closer together, $|\vec{u}|$ is bigger. The volume flux through any curve from $\vec{x_0}$to $\vec{x_1}$ is given by $\int_{x_0}^{x_1}\vec{u}\cdot\vec{n}dS=\psi(x_1)-\psi(x_0)$

Summarizing, $\frac{D}{Dt}=\frac{\partial}{\partial t}+\vec{u}\cdot\nabla$, $\frac{D}{Dt}\rho+\rho\nabla\cdot\vec{u}=\vec{0}$, $\vec{u}\cdot\vec{n}=\vec{U}\cdot\vec{n}$ and $\nabla\cdot\vec{u}=0$. In 2D cartesians this last condition gives $\vec{u}=(\psi_y,-\psi_x)$; in 2D polars, $u_r=\frac{1}{r}\frac{\partial}{\partial\theta}\psi, u_\theta=-\frac{\partial}{\partial r}\psi$, in axisymmetric cylindrical polars (i.e. $u_\theta=\partial_\theta\equiv 0$) $u_r=-\frac{1}{r}\frac{\partial}{\partial z}\Psi, u_z=\frac{1}{r}\frac{\partial}{\partial r}\Psi$ and in axisymmetric spherical polars ($u_\phi=\partial_\phi\equiv 0$) $u_r=\frac{1}{r^2 S_\theta}\partial_\theta\Psi, u_\theta=\frac{1}{r^2 S_\theta}\partial_r\Psi$ where S_θ means $\sin\theta$.

Note that if $\vec{u}=\nabla\times\vec{A}$ $\vec{A}$ is a vector potential and we have $\nabla\cdot\vec{u}=0$ automatically; this is the case for these examples, in the first two $\vec{A}=(0,0,\Psi)$, in the last two $\vec{A}=\left(0,\frac{\Psi}{r},0\right)$.

Motion of a material line elt

A material line elt (or curve) is a small line elt (or curve) of fluid material, i.e. it moves with the fluid. Say we have a small line segment $\delta\vec{l}$, with the endpoints initially at $\vec{x},\vec{x}+\delta\vec{l}$; then in time δt, the first endpoint moves to $\vec{x}+\vec{u}(\vec{x})\delta t$ but the second moves to $\vec{x}+\delta\vec{l}+\vec{u}\left(\vec{x}+\delta\vec{l}\right)\delta t$ (neglecting higher order terms), so $\delta\vec{l}$ becomes $\delta\vec{l}+\left(\vec{u}\left(\vec{x}+\delta\vec{l}\right)-\vec{u}(\vec{x})\right)\delta t=\delta\vec{l}+\left(\left(\delta\vec{l}\cdot\nabla\right)\vec{u}\right)\delta t$ i.e. $\frac{D}{Dt}\delta\vec{l}=\left(\delta\vec{l}\cdot\nabla\right)\vec{u}$, or $\frac{D}{Dt}\delta l_i=\delta l_j\partial_j u_i$; $\frac{\partial u_i}{\partial x_j}$ called the velocity gradient tensor. We can interpret this as saying that the rate of change of a line element (following the motion of the fluid) is either proportional to the change of velocity in the direction of the line element, or proportional to the velocity gradient tensor.

There are two types of forces on fluids; volume forves, where the force on a volume elt δv is some $\vec{F}(\vec{x},t)\,\delta v$; these are often conservative with PE some χ per unit volume so $\vec{F} = -\nabla\chi$ or some ϕ per unit mass so $\vec{F} = -\rho\nabla\phi$, e.g. gravity has $\phi = -gz$ so $\vec{F} = \rho\vec{g}\delta v$; $\rho\delta v$ is the mass of δv.

Surface forces have the force on a surface elt with unit normal $\vec{n}$ is $\vec{F}(\vec{x},t,\vec{n})\,\delta A$; we ignore friction and then the surface force is $\perp$ the surface with magnitide indep of orientation; normally $\vec{F}\delta a = -p(\vec{x},t)\,\vec{n}\delta A$ the pressure.

Momentum Equations

These are really simply conservation of momentum

Momentum Integral

For an arbitrary volume V fixed in space w/ sufficiently smooth surface ∂V and outward unit normal $\vec{n}$ the momentum inside V is $\int_V \rho\vec{u}dV$. This changes due to 3 processes: momentum flux across the boundary, surface forces and volume forces. The momentum flowing out of dS in time δt will be $\rho\vec{u}\cdot\vec{n}dS\vec{u}\delta t$; we have $\frac{d}{dt}\int_V \rho\vec{u}dV = -\int_{\partial V}\rho\vec{u}\,(\vec{u}\cdot\vec{n})\,dS - \int_{\partial V} p\vec{n}dS + \int_V \vec{F}dV$ where p is the surface force (pressure) per unit area and $\vec{F}$ the body force per unit volume e.g. gravity ρg; this equation is the equivalent of $\vec{F} = m\vec{a}$ for fluids; also expressed as $\frac{d}{dt}\int_V \rho u_i dV = -\int_{\partial V}\rho u_i u_j n_j dS - \int_{\partial V} p n_i dS + \int_V F_i dV$; $\rho u_i u_j$ is called the momentum flux tensor. We use the div thm: $\int_V \frac{\partial}{\partial t}\rho u_i dV = -\int_V \frac{\partial}{\partial x_j}\rho u_i u_j dV - \int_V \frac{\partial p}{\partial x_i}dV + \int_V F_i dV$; shrinking V to a point this becomes $u_i\left(\frac{\partial\rho}{\partial t} + \frac{\partial}{\partial x_j}(\rho u_j)\right) + \rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + F_i$; the first term is 0 by the incompressibility condition so $\rho\left(\frac{\partial u_i}{\partial t} + u_j\frac{\partial u_i}{\partial x_j}\right) = -\frac{\partial p}{\partial x_i} + F_i$ and $\rho\frac{D\vec{u}}{Dt} = -\nabla p + \vec{F}$; this is Euler's Equation

An alternate, more heuristic derivation comes from considering a small cuboid of fluid $\delta x_1 \times \delta x_2 \times \delta x_3$ as though it were a particle; use $m\vec{a} = \vec{F}$. The pressure on the two faces $\perp$ the x_1 axis is p_1 and $p_1 + \frac{\partial p_1}{\partial x_1}\delta x_1$; the mass of the fluid is $\rho\delta x_1\delta x_2\delta x_3$ so $\rho\delta x_1\delta x_2\delta x_3\frac{Du_1}{Dt} = -\frac{\partial p_1}{\partial x_1}\delta x_1\delta x_2\delta x_3 + F_1\delta x_1\delta x_2\delta x_3$ as the pressures are acting on faces of area $\delta x_2\delta x_3$; sim for the other cpts gives the equation as above. If we consider viscosity there is a $\mu\nabla^2\vec{u}$ term on the RHS, which makes the equation much harder to solve as it becomes second order.

Applications of momentum integral

The momentum integral sometimes enables us to gain useful results where calculating the complete fluid motion is impossible; such things are frequently the case in fluid mechanics.

In general pressure increases when velocity decreases and vice versa

Bernoulli's Streamline Thm

For steady flows with potential forces, $\rho\left(\frac{\partial\vec{u}}{\partial t}+(\vec{u}\cdot\nabla)\vec{u}\right)=-\nabla(p+\chi),\vec{F}=-\nabla\chi$. $\frac{\partial\vec{u}}{\partial t}=0$ so $\rho(\vec{u}\cdot\nabla)\vec{u}=-\nabla(p+\chi)$. Recall from VC $\vec{u}\times(\nabla\times\vec{u})=\nabla\left(\frac{1}{2}u^2\right)-(\vec{u}\cdot\nabla)\vec{u}$ [I shall omit $\vec{}$ for vectors from now on except where ambiguity would result]; let $\nabla\times u=\omega$ be the vorticity; ρ is constant so $\rho\vec{u}\times\vec{\omega}=\nabla\left(\frac{1}{2}\rho u^2+p+\chi\right)$; we define $H=\frac{1}{2}\rho u^2+p+\chi$; this is the energy per unit volume; the first term is KE, the second the pressure energy or compressibility and χ is the potential.$(\vec{u}\cdot\nabla)H=\rho\vec{u}\cdot\vec{u}\times\vec{\omega}=0$ so H is constant along streamlines; similarly $(\vec{\omega}\cdot\nabla)H=0$, H is const along vortex lines which are lines tang to small vecs representing the $\vec{\omega}$ field.

Applications

Emptying a container from a small hole in the bottom - we find the outward v satisfies $\frac{1}{2}v^2=gh$ where h is the height of liquid in the container, which we expect since this means its KE is equal to the lost PE.

A Pitot tube is a small tube used for measuring airspeed; at the inner end we have $u\approx 0$ and pressure p_1, while at the outer end fluid is flowing in at some velocity U and at atmospheric pressure p_a. We have $\frac{1}{2}\rho_a U^2+p_a=p_1$ so we can find U by $\sqrt{\frac{2(p_1-p_a)}{\rho_a}}$.

Using a venturi meter to measure flow in pipes - we attach a small open pipe to the top of our pipe and measure the height h of fluid in the small pipe supported by the pressure in the larger pipe. At some point the pipe has flow velocity U_1, cross section A_1 and pressure p_1; then further allong the pipe narrows to smaller cross section A_2, so we expect the fluid here to have greater velocity U_2 and therefore lower pressure p_2. Applying Bernoulli we have $\frac{1}{2}\rho U_1^2+p_1=\frac{1}{2}\rho U_2^2+p_2$. By mass conservation we have $A_1U_1=A_2U_2$, so $p_1-p_2=\frac{1}{2}\rho U_1^2\left(\frac{A_1^2}{A_2^2}-1\right),>0$, if we measure at both points this will be $\rho g\delta h$ so we can measure this δh to find U_1.

A 2D (horizontal) liquid jet hitting an inclined plane (at angle β from the horizontal, so that for $\beta<\frac{\pi}{2}$ the bottom points towards the jet) at speed U with initial cross section area a and output area a_1 at the bottom, a_2 at the top. We neglect gravity; the pressure on the free surface at the edge of the jet must be p_a; the velocity must therefore remain U on the free surface by Bernoulli, so we have the output velocity being U at both ends. Then by mass cons $a=a_1+a_2$ and by steady momentum flux $\parallel$ the plane $\rho aU^2c_\beta=\rho a_2U^2-\rho a_1U^2$ so $a_1=\frac{1}{2}a(1-c_\beta),a_2=\frac{1}{2}a(1+c_\beta)$; note both these are ≥ 0. $\perp$ the plane ρaU^2s_β is the force on the plane per unit $\perp$ distance; if we consider the couple we find the force moves the plane so it is $\perp$ the stream.

Further examples are an aerofoil, where we have velocity along the top surface $u_+ > u_-$ on the bottom surface. so $p_+ < p_-$, or the situation where two ships approach closely alongside each other while moving in the same direction; we get a larger velocity U through the gap so a smaller pressure, resulting in a force pulling them together. Likewise, as seen on the example sheet, a barge can ground on a deeper bottom than we would expect because when the gap between the barge and the bottom becomes narrow the fluid speed through it increases so the pressure decreases. A vacuum pump works by passing a high velocity past the end of a tube creating a low pressure which sucks fluid out of said tube. Stability of a table tennis ball in a uniform flow should be investigated by the reader. Finally if $\vec{u} = 0$ then $p + \chi$ is constant everywhere, so under gravity $p = p_0 - \rho g z$ where z is height - there is a linear decrease of pressure with height due to the weight of the overlying fluid.

Alternatively, we have the local eqn $0 = -\nabla p - \rho g \widehat{\vec{z}}$; taking cpts, $p_z = p_0 - \rho g z, p_x = p_y = 0$. Consider the pressure force on an arbitrary volume V in fluid of density ρ: $\vec{F} = -\int_{\partial V} (p_0 - \rho g z) \vec{n} dS$ which by the div thm is $-\int_V \nabla \cdot (p_0 - \rho g z)$ which is $\rho g V \widehat{\vec{z}}$. This is of course the weight of displaced fluid.

2.6 Vorticity

This is like angular momentum for fluid parcels; we have already defd $\vec{\omega} = \nabla \times \vec{u}$ or $\omega_i = \epsilon_{ijk} \frac{\partial}{\partial x_j} u_k$. In cartesians this gives $\vec{\omega} = (w_y - v_z, u_z - w_x, v_x - u_y)$ where $\vec{u} = (u, v, w), u_z = \frac{\partial u}{\partial z}$ etc.

Consider rigid body rotation of a fluid: $\vec{u} = \vec{\Omega} \times \vec{x}$ where $\vec{\Omega}$ is the angular velocity vector [I think]; then $\omega_i = \epsilon_{ijk} \partial_j u_k = \epsilon_{ijk} \partial_j \epsilon_{klm} \Omega_l x_m = \Omega_i \frac{\partial x_j}{\partial x_j} - \Omega_j \frac{\partial x_i}{\partial x_j} = 3\Omega_i - \Omega_i = 2\Omega_i$ so $\vec{\Omega} = 2\vec{\Omega}$; compare this with $\nabla \times$ the Euler eqn, which gives $\nabla \times \left(\frac{\partial \vec{u}}{\partial t} - \vec{u} \times \vec{\omega}\right) = -\nabla \times \nabla \left(\frac{p}{\rho} + \chi + \frac{1}{2} u^2\right)$ i.e. $\frac{\partial \vec{\omega}}{\partial t} - \nabla \times (\vec{u} \times \vec{\omega}) = 0$; $\nabla \times (\vec{u} \times \vec{\omega}) = (\vec{\omega} \cdot \nabla) \vec{u} - \omega (\nabla \cdot \vec{u}) + \vec{u} (\nabla \cdot \vec{\omega}) - (\vec{u} \cdot \nabla) \vec{\omega}$ [the reader should check this as I haven't]; $\nabla \cdot \vec{\omega} = \nabla \cdot \nabla \times \vec{u} = 0$ and $\nabla \cdot \vec{u} = 0$ so this is $(\vec{\omega} \cdot \nabla) \vec{u} + (\vec{u} \cdot \nabla) \vec{\omega}$ and substituting this back in we have $\frac{D\vec{\omega}}{Dt} = (\vec{\omega} \cdot \nabla) \vec{u}$; this is similar to the motion of a material line elt $\frac{D}{Dt} \delta \vec{l} = \left(\delta \vec{l} \cdot \nabla\right) u$; we shall see more of this later.

In suffix notation this is $\frac{D\omega_i}{Dt} = \omega_j \frac{\partial u_i}{\partial x_j}$; $\frac{\partial u_i}{\partial x_j}$ is the velocity gradient tensor; we express it in symmetric and antisymmetcir parts as $\frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i}\right) + \frac{1}{2}\left(\frac{\partial u_i}{\partial x_j} - \frac{\partial u_j}{\partial x_i}\right)$ and call this $e_{ij} + \frac{1}{2} \epsilon_{jik} \omega_k$(yarly, check it if you don't believe me) where e_{ij} is called the pure strain. The vortex line elts move as if they were material line elts, i.e. vortex lines move with the fluid; like everything in IB this depends on the assumptions that ρ is constant and $\mu \equiv 0$. For example, for a rotating cylindrical small elt if the cylinder is stretched, $\vec{\omega}$ is axial so also stretched in direct proportion; this is rise to the "ballerina effect" and can be seen in e.g. a plughole vortex. In rotating systems like the Earth it is very important e.g. for modelling the weather.

2.7 Kelvin's Circulation Thm

This is an integral form of the vorticity eqn: define circulation $C(t)$ for a closed curve $\Gamma(t)$ (which moves with the fluid, i.e. a material curve) by $C(t) = \oint_\Gamma \vec{u}\cdot d\vec{l}$ (note this is $\int_S \vec{\omega}\cdot\vec{n}dS$ for a capping surface S by Stokes' Thm); $\frac{dC}{dt} = \oint_\Gamma \frac{D\vec{u}}{Dt}\cdot d\vec{l} + \oint_\Gamma \vec{u}\cdot\frac{D\delta\vec{l}}{Dt} = \oint_\Gamma -\nabla\left(\frac{p+\chi}{\rho}\right)\cdot d\vec{l} + \oint_\Gamma \vec{u}\cdot\left(\delta\vec{l}\cdot\nabla\right)\vec{u}d\vec{l}$ [check intermediate steps; result is correct]; the last term is $\oint_\Gamma \frac{1}{2}\nabla\vec{u}^2\delta\vec{l}d\vec{l}$ so this is $\left[\frac{1}{2}u^2 - \frac{(p+\chi)}{\rho}\right]$ which is 0 as Γ is closed. In particular if $C = 0$ everywhere at $t = 0$, $C = 0$ forever and letting Γ shrink to a point if $\vec{\omega} = 0$ at $t = 0$ $\vec{\omega} = 0$ forever. This divides fluid mechanics into 2 sections: $\vec{\omega} = 0$ i.e. $\nabla\times\vec{u} = 0$, called irrotational flow, which is essentially a closed field, and $\vec{\omega} \neq 0$ called rotational flow which is much harder.

Irrotational flows are called potential flows because if $\nabla\times\vec{u} = 0$ $\vec{u} = \nabla\phi$ for some potential ϕ. A flow initially irrotational everywhere remains irrotational everywhere; this situation is quite common as it occurs when we start from rest or if we have uniform (so irrotational) flow coming from upstream. We have $\vec{u} = \nabla\phi + f(t)$; we can find ϕ by reversing this $\phi(\vec{x_1}, t) = \int_{x_0}^{x_1}\vec{u}(\vec{x}, t)\cdot d\vec{l}$. Mass conservation $\nabla\cdot\vec{u} = 0$ gives $\nabla^2\phi = 0$ and we have our boundary cond $\vec{U}\cdot\vec{n} = \vec{u}\cdot\vec{n} = \vec{n}\cdot\nabla\phi = \frac{\partial\phi}{\partial\vec{n}}$, so we simply need to solve Laplace's eqn (which is linear) with von Neumann BCs

Examples

There are many types of sol to this eqn, which correspond to different problems. In axisymmetric spherical polars the general sol is $\phi = \sum_{n=0}^{\infty}\left(A_n r^n + B_n r^{-n}\right)P_n(\cos\theta)$ where P_n are the Legendre functions or polynomials $P_0 = 1, P_1 = \cos\theta, P_2 = \frac{1}{2}\left(3\cos^2\theta - 1\right), \ldots$; in general we will only use these first few modes, e.g. $\phi = -\frac{m}{4\pi}\frac{1}{r}$ giving $\vec{u} = \nabla\phi = \frac{m}{4\pi}\frac{\hat{r}}{r^2}$, radial flow $\propto \frac{1}{r^2}$ [$\hat{r}$ being a unit vector in the direction of $\vec{r}$]; the flow out of a sphere of radius R is $4\pi R^2 u_R = m$, indep of R as we would expect from mass cons; this is a point source of strength m (or point sink for $m < 0$). $\phi = U_r\cos\theta \equiv Uz$ gives $\vec{u} = \nabla\phi = U\hat{z}$, uniform flow; likewise in cartesians $\phi = \vec{U}\cdot\vec{x}$ gives $\vec{u} = \vec{U}$ constant.

In 2D polar or cylindrical geometry the general sol is $\phi = K + A_0\ln r + B_0\theta + \sum_{n=1}^{\infty}\left(A_n r^n + B_n r^{-n}\right)e^{in\theta}$; this can of course be expressed in sins and coses for A_n, B_n real. For example $\phi = \frac{m}{2\pi}\ln r$ gives $u = \frac{m}{2\pi}\frac{\hat{r}}{r}$; radial flow $\propto \frac{1}{r}$, and the flow out of a circle of raduis R is $2\pi Ru_R = m$; this is a line or 2D point source of strength m. Or $\phi = \frac{\kappa}{2\pi}\theta$ gives $\vec{u} = \frac{\kappa}{2\pi}\frac{\hat{\theta}}{r}$, circilar flow $\propto \frac{1}{r}$ with the circulation around a circle of radius R being κ, independent of R since $\nabla\times\vec{u} = 0$; this is a line vortex of circulation κ.

If we consider uniform flow coming in at speed U from $-\infty$ and flowing past a sphere of radius a, taking $\theta = 0$ to be the direction of the incoming flow. There is no vorticity in the incoming uniform flow so $\nabla^2\phi = 0$ for $r > a$; the flow should be uniform at ∞ so $\phi \to urc_\theta$ as $r \to \infty$, and $\frac{\partial\phi}{\partial r} = 0$ at $r = a$

- there is no flow in or out of the sphere. Since the problem is linear the sol is unique [???]; we must have the sol taking the form $\phi = Uc_\theta\left(r + Br^{-2}\right)$; the BC at $r = a$ gives $1 - 2Ba^{-3} = 0$ so $\phi = Uc_\theta\left(r + \frac{a^3}{2r^2}\right)$; then $\vec{u} = \nabla\phi = \left(Uc_\theta\left(1 - \frac{a^3}{r^3}\right), -Us_\theta\left(1 + \frac{a^3}{2r^3}\right), 0\right)$ [the third coord being 0 by symmetry); this has $u_r = 0$ at $r = a$ satisfying the BC, and $u_\theta \neq 0$ as we would expect, since although this is not strictly a BC it would be odd to have flow completely still on the surface of the sphere.

If we draw the streamlines we find they are more crowded at the points B on the sphere where $\theta = \frac{\pi}{2}$, and the point $\theta = 0$ is in fact a stagnation point, though there is flow near it. Though beautiful this sol is entirely unlike RL; even a tiny nonzero viscosity drastically alters the solution, completely destroying the "front-back" symmetry of the streamlines amongst other things.

Next we consider the same cross-section as the above, but rather than a sphere we have a cylinder - we can just consider 2D flow past a disc. Again $\phi \to Urc_\theta$ as $r \to \infty$ and $\frac{\partial\phi}{\partial r} = 0$ at $r = a$. Then $\phi = Uc_\theta\left(r + \frac{a^2}{r}\right) + \frac{\kappa}{2\pi}\theta$ for arbitrary constant κ, giving $\vec{u} = \left(Uc_\theta\left(1 - \frac{a^2}{r^2}\right), -Us_\theta\left(1 + \frac{a^2}{r^2}\right) + \frac{\kappa}{2\pi r}\right)$; we find the line integral of $\vec{u}$ around a closed curve about the cylinder is κ, the circulation. It is essential to specify this if we want a unique solution; we cannot just assume it is 0.

3.3 Pressure in potential flows with conservative forces

$\rho\left(u_t + (\vec{u}\cdot\nabla)\vec{u}\right) = -\nabla(p + \chi)$; $(\vec{u}\cdot\nabla)\vec{u} = \nabla\left(\frac{1}{2}u^2\right) + \vec{\omega}\times\vec{u} = \nabla\left(\frac{1}{2}u^2\right)$ and $\frac{\partial u}{\partial t} = \nabla\left(\frac{\partial\phi}{\partial t}\right)$ so we have $\nabla\left(\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi\right) = \vec{0}$, everywhere, so $\xi(\vec{x}, t) = \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho u^2 + p + \chi = f(t)$ independent of $\vec{x}$ (generally we will ignore the time dependence and just use that $\xi(\vec{x_1}) = \xi(\vec{x_2})$ at fixed time. The reader should pay attention to some important points raised on the example sheets for this course.

As an example, consider the fast (i.e. we ignore gravity) jet generator; a large container from $x = -\xi$ to $x = 0$ discharging via a thin tube from $x = 0$ to $x = l$ into an atmosphere at p_a. We have $p(x = 0) = p_0(t) + p_a$ for some function p_0; the fluid starts from rest so it is irrotational so velocity potential. We assume the flow in the thin tube is uniform. $\vec{u} = (U(t), 0, 0)$ gives $\phi = U(t)\,x$ $(+h(t)$ but we neglect this term as it does not affect $\vec{u}$); we can now compare $f(t)$ at $x = 0, l$; the reader should do this as an exercise.

$\rho\dot{U}0 + \frac{1}{2}\rho U^2 + p_0(t) + p_a = \rho\dot{U}L + \frac{1}{2}\rho U^2 + p_a$ so $\dot{U} = \frac{p_0(t)}{\rho L}$. $u(t) = \frac{1}{\rho L}\int_0^t p(s)\,ds = \frac{p_\star t}{\rho L}$ where $p_\star = p_0(t)$.

Next we consider the fast jet generator with conditions at $x = -\xi$, namely pressure $p_1(t) + p_a$. We have $\phi \approx 0, u \approx 0$ here since the container is large, so $p_1(t) + p_a = \rho\dot{U}L + \frac{1}{2}\rho U^2 + p_a$ [I am unsure of the case of many of the Us in this lecture], a first order nonlinear DE for U. Consider $p_1(t) = p_2$ constant for $t > 0$; when $t = 0$ $\rho\dot{U}L + \frac{1}{2}\rho U^2 = p_2$ w/ $u = 0$ when $t = 0$. We nondimensionalize by considering a velocity scale $\frac{\sqrt{2p^2}}{\rho} = U_0$[I am unsure as to whether ρ is also square rooted] and time scale $\frac{2L}{U} = t_0$; we consider non-dimensional variables $\eta = \frac{u}{u_0}, \tau = \frac{t}{t_0}$ and the eqn becomes $\frac{d\eta}{d\tau} = 1 - \eta^2, \eta(0) = 0$; the sol is $\eta = \tanh\tau$ or $u = u_0 \tanh\left(\frac{u_0 t}{2L}\right)$; this technique is more useful when we cannot solve our equations or they are not accurate enough, since it indicates to us what the length and time scales are.

Free oscillations of a manometer

This is a tube (of constant cross section for now) curved into a U shape. Say external pressure is p_a, use an arclength coordinate s; $s = l + h(t)$ at one end of the fluid and $-l + h(t)$ at the other where l is the undisturbed length, both ends being at angle α above the horizontal; we must consider gravity g since it is the cause of oscillation. The fluid starts from rest so $\nabla^2\phi = 0$; We use unsteady bernoulli. $\phi = \dot{h}s, \frac{\partial\phi}{\partial t} = \ddot{h}s, u = \dot{h}$. We compare the two surface points; $\rho\ddot{h}(-l+h) + \frac{1}{2}\rho\dot{h}^2 + p_a - \rho g s_\alpha(-l+h) = \rho\ddot{h}(l+h) + \frac{1}{2}\rho\dot{h}^2 + p_a + \rho g s_\alpha(l+h)$ [sic; not necessarily correct] so $l\ddot{h} + g s_\alpha h = 0$ giving SHM w/ freq $\sqrt{\frac{g s_\alpha}{l}}$; this agrees surprisingly well with experiment, viscosity does not affect the frequency significantly, just damps the oscillations. Note:

1. Since we compared only the two ends, what happens away from the free surfaces is irrelevant, e.g. the cross section in the middle of the tube may change
2. The reader should attempt to extend this to the situation where the two ends are at different angles
3. A harder problem: vary the cross-sectional area at the two ends. This and the previous problem give a nonlinear eqn.

3.6 Bubbles

3.6.1 General Theory for spherically symmetric motions

It is often appropriate to assume bubbles are spherical, though this does not work for big bubles (since their shape is altered by instability and their wakes) or strange fluids. Surface tension helps keep the bubble spherical but does not affect its dynamics, as will be seen in the part II course nonlinear dynamics.

Say the radius of the bubble is a, in a liquid of pressure $p(x,t)$ (we consider only radial x since everything is spherically symmetric) where $p(\infty, t)$ is given as a func of time; we seek $p(a,t)$ and a as a function of time; $u(r) \propto \frac{1}{r^2}$ since $\nabla \cdot u = 0$ in the liquid; $\int ur^2 dr$ must be the same over a sphere of any rad R about the origin. We neglect gravity as it is generally uninfluential; $u(r=a) = \dot{a}$ so $\vec{u} = \frac{\dot{a}a^2}{r^2}\hat{r}$ meaning $\phi = -\frac{\dot{a}a^2}{r}$; $\frac{\partial\phi}{\partial t} = -\frac{\ddot{a}a^2 + 2a\dot{a}^2}{r}$; comparing at $r = a, \infty$ $-\rho\frac{\left(\ddot{a}a^2 + 2a\dot{a}^2\right)}{r} + \frac{1}{2}\rho\dot{a}^2\frac{a^4}{r^4} + p(r,t) = p(\infty, t)(1)$; on $r = a$ $p(\infty,t) - p(a,t) = -\rho\ddot{a}a - \frac{3}{2}\rho\dot{a}^2(2)$ (We could differentiate between $p(a+,t)$ and $p(a-,t)$

due to surface tension but will not in this course); Multiplying this by $a^2\dot{a}$ we have $\frac{d}{dt}\left(\frac{1}{2}\rho a^3\dot{a}^2\right) = (p_a - p_\infty)\, a^2\dot{a}$ (3); the way to interpret this is that $\frac{d}{dt}(KE)$ = rate of working by pressure at $r = a$ on the whole liquid, since $KE = \frac{1}{2}\rho\int_a^\infty u^2 dV = \frac{1}{2}\rho\int_a^\infty \frac{\dot{a}^2a^4}{r^4}4\pi r^2 dr = 2\pi\rho\dot{a}^2a^3$ [lower limit is 0 in my notes], and rate of working is force×velocity or $\delta p 4\pi a^2 \times \dot{a}$ where $\delta p = p_a - p_\infty$. Then eliminating $\ddot{a}$ between eqns 1,2 we have $p(r,t) = p(\infty,t) + (p(a,t) - p(\infty,t))\frac{a}{r} + \frac{1}{2}p\dot{a}^2\left(\frac{a}{r} - \frac{a^4}{r^4}\right)$; we call the last term here T_2 and the other two terms T_1. For $p_\infty > p_a$ we have T_1 being p_a at $\frac{r}{a} = 1$ and tending monotonically but at gradually decreasing rate up to p_∞ as $\frac{r}{a} \to \infty$, so $\frac{\partial T_1}{\partial r} < 0$ meaning $\frac{Du}{Dt} < 0$; we have decelleration. The reader should consider the case $p_a > p_\infty$. If we plot $\frac{2T_2}{\rho\dot{a}^2}$ we have a "bump" rising from 0 at $\frac{r}{a} = 1$ to a peak at $4^{\frac{1}{3}} \approx 1.58$ and then decaying exponentially; we have decelleration for $\frac{r}{a} < 4^{\frac{1}{3}}$, accelerating for $\frac{r}{4} > 4^{\frac{1}{3}}$.

3.6.2 Small oscillations of a gas bubble

$a(t) = a_0 + \eta(t)$ with $|\eta| << a_0$; we are considering rapid adiabatic oscillations of air in a bubble, which satisfy the gas law that pv^γ is constant where γ is the ratio of specific heats. $\frac{\delta p}{p} = -\gamma\frac{\delta v}{v}$ which is $-3\gamma\frac{\eta}{a_0}$ since $v \propto a^3$. If we linearise eqn 2 we have $\ddot{\eta} + \left(\frac{3\gamma p_\infty}{\rho a_0^2}\right)\eta = 0$ so we have SHM w/ freq $\sqrt{\frac{3\gamma p_\infty}{\rho a_0^2}}$; this is $\sim 2\times 10^4 s^{-1}$ for a 1mm bubble and $2\times 10^2 s^{-1}$ for a 1cm bubble.

3.6.3 Collapse of cavities

In many flows around bodies e.g. propellors U increases to the extent that mathematically we would expect p to decrease to below 0. Physically this is impossible; instead a bubble filled with vapour forms, with pressure << that at ∞, which will therefore rapidly collapse.

Consider a spherical cavity at zero internal pressure $p_a = 0$ initially at rest ($\dot{a} = 0$ when $t = 0$) in constant background pressure $p_\star$; applying 3 above we have $\frac{1}{2}\rho\frac{d}{dt}\left(a^3\dot{a}^2\right) = -p_\star a^2\dot{a}$; taking an integral $\frac{1}{2}\rho a^3\dot{a}^2 = \frac{1}{3}p_\star\left(a_0^3 - a^3\right)$ (where $a_0 = a(0)$) so $\frac{\dot{a}}{\sqrt{\frac{a_0^3}{a^3}-1}} = -\sqrt{\frac{2}{3}\frac{p_\star}{\rho}}$ (4) (we take the −ve root since we know from the physics the cavity will be shrinking); notice there is a singularity as $a \to 0$ with $\dot{a} \propto a^{-\frac{3}{2}}$; there are huge velocities as the cavity collapses. 4 can only be integrated numerically, but the reader can see that there will be some finite collapse time and should be able to derive that this time is $\propto \sqrt{\frac{\rho a_0^2}{p_\star}}$; in fact the constant of proportionality is 0.92. As $a \to 0$ the maximum pressure is $\sim \dot{a}^2 \sim a^{-3}$, e.g. $p_{\text{water}} \approx 10^3 kgm^{-3}, p_\star = 1atm \approx 10^5 Pa$; when $a = \frac{1}{10}a_0$ $\dot{a} \approx 260ms^{-1}, p_{\max} \approx 160atm$; this effect is known to melt propellors.

3.7 Translating Accelerating Spheres

Say we have a uniformly moving sphere; relative to the sphere fluid is moving at speed U in the $\theta = 0$ direction; we have $\phi = Uc_\theta \left(r + \frac{a^3}{2r^2}\right), \vec{u} = \nabla\phi = \left(Uc_\theta \left(1 - \frac{a^3}{r^3}\right), -Us_\theta \left(1 + \frac{a^3}{2r^3}\right), 0\right)$ so u on $r = a$ is $\frac{3}{2}Us_\theta$ [in the radial direction]. $\partial_t \equiv \chi \equiv 0$ so we can find the pressure by $\frac{1}{2}\rho\left(\frac{3}{2}Us_\theta\right)^2 + p(a,\theta) = \frac{1}{2}\rho U^2 + p_\infty$ giving $p(a,\theta) = p_\infty + \frac{1}{2}\rho U^2 \left(1 - \frac{9}{4}s_\theta^2\right)$; as we would expect the pressure is large at $\theta = 0, \pi$ and small at $\theta = \pm\frac{\pi}{2}$. However, the symmetry of pressure (fore/aft and sideways) means that in potential flow there can be no drag or lift on a sphere; this is clearly false IRL, a problem made worse by D'Alembert's paradox: on a general 3D body there can be no drag (i.e. force $\parallel$ the stream) in a uniform flow; the "solution" to this is of course that viscosity is critically important in these problems and cannot be neglected. The part II course returns to this problem.

However, potential flow works well for bubbles as above, and this example which is probably its greatest success:

3.4 Flows w/ free surface

The usual example of these is an air/water interface; we have $\rho_a << \rho_w$, and often put $p = p_a$ at the interface. This is useful for modelling water waves, tides, floods on rivers and over land, control of weirs and more.

Governing Eqns

We assume the flow is irrotational (say motion starts from rest) so we have potential flow: $\vec{u} = \nabla\phi, \nabla^2\phi = 0$; the dynamics are that $\rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left|\nabla\phi\right|^2 + p + \rho gz = f(t)$ where z is the vertical axis. We ignore surface tension; it can play a role in this situation but doesn't for larger waves, in the case of air/water those of wavelength $>\sim 23cm$; so we set the pressure on the surface to be p_a and then $\rho\frac{\partial\phi}{\partial t} + \frac{\rho}{2}\left|\nabla\phi\right|^2 + pgz = f_1(t)$ (we have absorbed p_a into f) on $z = \xi(x, y, t)$ where this is the height of the surface. The kinematics are that $\frac{\partial\xi}{\partial t} + u\frac{\partial\xi}{\partial z} + v\frac{\partial\xi}{\partial y} = w = \frac{\partial\phi}{\partial z}$ on $z = \xi$ [At this point I cease to understand the lectures and merely transcribe].

3.4.2 Linear water waves

We have a see surface at undisturbed level $z = 0$ with the ocean floor or similar at $z = -h$; the full nonlinear problem is $\nabla^2\phi = 0$ for $-h \le z \le \xi$ with BC $\frac{\partial\phi}{\partial z} = 0$ on $z = -h$ (there is no vertical flow through the sea bed); $\frac{\partial\xi}{\partial t} + \frac{\partial\phi}{\partial x}|_\xi \frac{\partial\xi}{\partial x} + \frac{\partial\phi}{\partial y}|_\xi \frac{\partial\xi}{\partial y} = \frac{\partial\phi}{\partial z}, \rho\frac{\partial\phi}{\partial t} + \frac{1}{2}\rho\left|\nabla\phi\right|^2 + \rho g\xi = f(t)$ independent of x, y on $z = \xi$. We linearize; we assume $\xi << h, \frac{\partial\xi}{\partial x}, \frac{\partial\xi}{\partial y} << 1$; amplitude a is $<<$ wavelength λ. This is approprate to many situations; on the cam, $a \sim 2cm$ and $\lambda \sim 1m$ while in the ocean $a \sim 2m$ but $\lambda \sim 100m$. We therefore discard all quadriatically small terms and apply the BCs on $z = 0$ rather than ξ i.e. we are using the

Taylor series $\frac{\partial\phi}{\partial z}\mid_{z=\xi}=\frac{\partial\phi}{\partial z}\mid_{z=0}+\xi\frac{\partial^2\phi}{\partial z^2}\mid_{z=0}+\ldots$ and then discard all but the first term of this.

The problem is now $\nabla^2\phi=0$ on $-h\le z\le 0$, $\frac{\partial\xi}{\partial t}=\frac{\partial\phi}{\partial z}$ on $z=0$ and $\frac{\partial\phi}{\partial t}+gz=f(t)$ on $z=0$ (we have divided through by ρ and absorbed this into f)

Sols

There are three methods to find sols to this:

1. Look for seperable sols $\phi(x,y,z,t)=Z(z)Y(y)X(x)T(t)$; then $X(x)=e^{ikx}, Y(y)=e^{ily}, T(t)=e^{-i\omega t}$.

2. A better approach; use the fact that this is a linear problem and invariant wrt translations in x,y,t

3. Use physics; take the 2D case so $\partial_y=0$; $\xi(x,t)=ae^{i(kx-\omega t)}$ but ξ is real so must be $\Re\left(ae^{i(kx-\omega t)}\right)=a_1\cos(kx-\omega t+\psi)$ for some phase ψ. $\phi(x,z,t)=S(z)e^{i(kx-\omega t)}$ where S is some structure function; note this is one fourier mode. The real elevation $E(x,t)=\int_{-\infty}^{\infty}a(k)e^{i(kx-\omega(k)t)}dk$; we seek $S(z),\omega(k)$ and an eigenval or dispersion relationship. $\partial_t=i\omega,\partial_x=ik,\partial_{xx}^2=-k^2,\nabla^2=\partial_{zz}^2-k^2$ so $S''-k^2S=0$ on $-h<z<0$, $S'=0$ on $z=-h$ and $-i\omega\xi=\frac{\partial\phi}{\partial z}$ on $z=0$ where S is a pure func of z. We must have $f(t)=0$ since S is not a func of x so $-i\omega\phi+g\xi=0$ at $z=0$. This is a second order DE with 3 BCs, but these are actually split BCs as there is some interdependence; $\omega(k)$ is the eigenval [?], $S=Ae^{kz}+Be^{-kz}$; $\phi_z=0$ on $z=-h$ so $S=A\cosh k(z+h)$; $\xi_t=\phi_z$ so $i\omega a=kA\sinh kh$ meaning $S(z)=\frac{-i\omega a\cosh k(z+h)}{k\sinh kh}$. $-i\omega\frac{a}{k}\coth kh+ga=0$ so $\omega^2=gk\tanh kh$; k is the wavenumber $\frac{2\pi}{\lambda}$. $c^2=\frac{\omega^2}{k^2k}$ so the wave goes like $e^{ik(x-ct)}$; c is the wave speed. $c^2=\frac{g}{k}\tanh kh$. Long waves have k small, short waves have it large (though once waves get too small surface tension becomes important and this model becomes less valid); however, longer waves have smaller freqency and higher speed and vice versa. $kh=\frac{2\pi h}{\lambda}$; for kh small $\omega\sim k^{\frac{1}{2}}$ and $c\sim k^{-\frac{1}{2}}$.

Special Cases

1. Deep water waves $kh>>1$ i.e. $\lambda<<h$ have $\tanh kh\sim 1$ so $\omega\approx\sqrt{gk}, c\approx\sqrt{\frac{g}{k}}$; note this speed is independent of h; short wavelength waves propagate independently of the water depth, e.g. in the atlantic the dominant wavelength is $\sim 400m$ (period $\sim 16s$) $<<h\sim 4km$ so $c=\frac{\lambda}{T}\approx 25ms^{-1}$; a storm will propagate less rapidly so the waves "bring notice". $\tanh\pi\approx 0.9963$ so $h>\frac{1}{2}\lambda\Rightarrow\omega^2=\frac{g}{k}$ to within 1%.

2. Shallow water waves $kh<<1$ i.e. $\lambda>>h$; $\omega\approx\sqrt{gh}k, c\approx\sqrt{gh}$ independent of wavelength e.g. flood waves on a river have $h\sim 2m, c\sim 4.5ms^{-1}$,

tidle waves, storm surges in shallow seas, or tsunamis which are virtually non-dispersive because their wavelengths are so long.

Sound waves are very linear; they have almost no dispersion from both freq and amplitude, in stark contrast to water waves. We can see ourselves as being very lucky in this, since speech and music would be impossible without it, or we can see it as an example of the anthropic principle.

Without proof until part II: phases move at phasespeed $c = \frac{\omega}{k}$ but groups of waves, and energy, move at a group vel $c_g = \frac{d\omega}{dk}$ which is generally $\neq \frac{\omega}{k}$; this corresponds with observation.

3.5.2 Partical Paths

$\phi = \Re\left(-\frac{i\omega a}{k}\frac{\cosh k(z+h)}{\sinh kh}e^{i(kx-\omega t)}\right)$ so $\vec{u} = \nabla\phi = \Re\left(\frac{a\omega}{\sinh kh}\left(\cosh k\left(z+h\right), -i\sinh k\left(z+h\right)\right)e^{i(kx-\omega t)}\right) = (u, w)$; u is in phase with ξ, w is out of phase by $\frac{\pi}{2}$ - horizontal vel on the surface is maximal at the peaks (and in the other direction at the troughs), vertical vel maximal at the undisturbed height. $u >> w$ if $kh << 1$ i.e. shallow water e.g. rivers; this case is covered above [allegedly]. When $kh >> 1$ i.e. deep water $u \sim w$.

We find the particle paths by integrating $\frac{d\vec{x}}{dt} = \vec{u}\left(\vec{x}, t\right)$; using linearization we integrate at the mean value (x_0, z_0); $x\left(t\right) = \Re\left(x_0 + \frac{ia\cosh k(z_0+h)e^{i(kx_0-\omega t)}}{\sinh kh}\right)$, $z\left(t\right) = \Re\left(z_0 + \frac{ia\sinh k(z_0+h)e^{i(kx_0-\omega t)}}{\sinh kh}\right)$ [if I ever write sh this is sinh; this was used in the lecture but I am avoiding it since it's incredibly lame]. So the particle paths are ellipses, with radii decaying exponentially with z; in deep water $kh >> 1$ these become circles while in shallow water $u >> w$ the horizontal displacement is Â» the vertical displacement.

Note $Ae^{i(kx-\omega t)}$ is a progressive wave moving to the right with speed $\frac{\omega}{k}$; $A\left(e^{i(kx-\omega t)} + e^{i(kx+\omega t)}\right)$ is a standing wave.

This section as a whole is perhaps the greatest success of irrotational inviscid fluid mechanics; however, a much harder and still essentially unsolved problem is how waves are generated by wind over the surface of the water.

3.5.3 Deep water standing waves in a box

Say we are restricted to $0 \leq x \leq a, 0 \leq y \leq b, z \leq 0$ [since linearizing]; $\xi\left(x, y, t\right)$ is the height of the free surface with undisturbed height $z = 0$; we look for linearized standing waves. $\nabla^2\phi = 0$ on this area, $\phi_x = 0$ on $x = 0, a$ (since there is no normal flow through the walls of the container), $\phi_y = 0$ on $y = 0, b$, $\nabla\phi \to 0$ as $z \to -\infty$, and $\xi_t = \phi_z$ and $\phi_t + g\xi = 0$ on $z = 0$. We try and find a seperable sol; the BCs imply one of the form $A\cos\frac{m\pi x}{a}\cos\frac{n\pi y}{b}e^{kz}e^{-i\omega t}$; $\nabla^2\phi = 0 \Rightarrow -\frac{m^2\pi^2}{a^2} - \frac{n^2\pi^2}{b^2} + k^2 = 0$; only discrete values of k are allowed; compare this with QM.

$\xi_t = \phi_z\mid_{z=0}$ so $\xi = \frac{iAk}{\omega}\cos\frac{m\pi x}{a}\cos\frac{n\pi x}{b}e^{-i\omega t}$; $-i\omega\xi = k\phi, \phi_t + g\xi = 0 \Rightarrow \omega^2 = gk$ as before, which we would expect since we have the same field eqn.

This is for where we have density 0 above the free surface, ρ below it, but density 0 below and ρ above gives the same eqn but $g \to -g$, so $\omega^2 = -gk, \omega = \pm i\sqrt{gk}$; the surface behaves like $e^{-i\omega t} = e^{\pm\sqrt{gk}t}$; the fluid disturbance increases exponentially with time. This is again as we would expect thinking about the physical situation; this is the Rayleigh-Taylor instability.

3.7.2 Accelerating motion of sphere

Take axes fixed in space; we have a sphere of radius a at position $\vec{x_0}(t)$, moving in a fied direction at speed $\vec{U}(t)$; for a general vector $\vec{x}$ define $\vec{r} = -(\vec{x} - \vec{x_0}) = \vec{x_0} - \vec{x}$.

$\nabla^2\phi = 0$ for $r \geq a$, $\nabla\phi \to 0$ as $r \to \infty$, $\frac{\partial\phi}{\partial r}\mid_t = \vec{U}\cdot\hat{r}$ on $r = a$, where $\hat{r} = \frac{\vec{x}-\vec{x_0}}{|\vec{r}|}$. There is no t dependence in the field eqn so we have an instantaneous response i.e. $\phi = -\frac{1}{2}|U|\frac{a^3}{r^3}\cos\theta = -\frac{\vec{U}(t)\cdot(\vec{x}-\vec{x_0})a^3}{2|\vec{x}-\vec{x_0}|^3}$; recall that $\vec{x_0}$ is a function of t. The eqn must hold when U =constant, but recall that then there is no force on the sphere; $\frac{\partial\phi}{\partial t}\mid_r = -\dot{\vec{U}}\cdot\vec{r}\frac{a^3}{2r^3} + \vec{U}\cdot\ldots$ where the second term contributes no force.